NONISOTHERMAL MOTION OF A GAS IN A CHANNEL WITH PARTIAL ACCOMODATION AT THE WALLS

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UDC 533.6.011.8

The presently available kinetic models of transport processes in gases usually give a sufficiently rigorous description of the collisions of gas molecules with each other, but the interaction of gas molecules with a solid surface is normally taken into account only under the rough assumption of specular or diffuse reflection. The gas–surface interaction is often simplified by modeling the distribution of molecules reflected by the surface as a power series in the velocity. The terms of the series are determined by introducing coefficients of accomodation for the various molecular characteristics. But the solution of the Boltzmann equation subject to boundary conditions of this kind, and therefore the model itself, contains an internal contradiction resulting from the fact that the kinetic processes occurring in the bulk region of the gas are described at a completely different level from processes occurring at the solid surface.

A systematic kinetic treatment of the motion of a gas in a channel requires a description of the motion of molecules in the region where the fluctuating surface force field acts. An appropriate kinetic equation for this problem and its solution were presented in [1]. These results give a relation between the distribution function of the molecules incident at the wall and that of the reflected molecules.

In the present paper we discuss a variational solution of the problem for the nonisothermal motion of a gas in an infinitely long channel for arbitrary density. The starting point is the S-model kinetic equation and boundary conditions formulated in terms of the scattering kernel. The Cercignani–Lampis scattering kernel [2] is chosen as a model for the numerical calculations. The dependence of the kinetic coefficients of the motion of the gas on the parameters describing the stochastic motion of gas molecules in the surface force field and on the gas–gas interaction is found.

1. We consider the steady motion of a single-component monatomic gas in an infinitely long cylindrical capillary of radius R. The motion is induced by longitudinal pressure

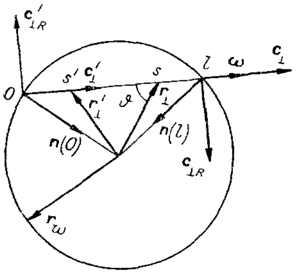

Fig. 1

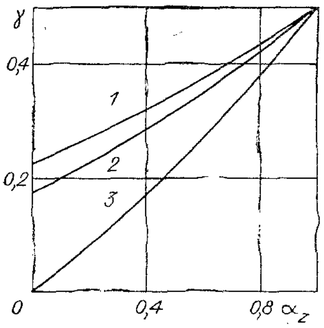

Fig. 2

Sverdlovsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 1, pp. 96-105, January-February, 1988. Original article submitted September 2, 1986.

and temperature gradients. The axis of the capillary is taken to be the z-axis. The surface of the capillary is smooth, and the gas is assumed to be weakly perturbed. The following quantities are chosen as scales of measurement in transforming the equations to dimensionless form: length, R; velocity $\beta_0^{-1/2} = (2kT_0/m)^{1/2}$; number density, n_0, temperature, T_0; pressure, p_0; distribution function, $\rho_0\beta_0^{3/2}/kT_0$ heat and particle flux densities $n_0\beta_0^{-1/2}$ and $kT_0n_0\beta_0^{-1/2}$, respectively, where k is the Boltzmann constant, and T_0, p_0, n_0 are certain average values of the temperature, pressure, and number density over the capillary. Since the motion of a gas in an infinitely long channel can be assumed to be translationally invariant, the distribution function of the gas molecules can be written as a Maxwellian distribution whose density and temperature are linear functions of z, plus a correction term that is independent of z: $f(\mathbf{r}_\perp, z, \mathbf{c}) = f^0(\mathbf{c})\{1 + z[\nu + \tau(c^2 - 5/2)] + h(\mathbf{r}_\perp, \mathbf{c})\}$, $f^0(\mathbf{c}) = \pi^{-3/2}\exp(-c^2)$. Here $\mathbf{r}_\perp$ is the two-dimensional position vector in the plane of the capillary cross section; ν and τ are the longitudinal pressure and temperature gradients; $h(\mathbf{r}_\perp, \mathbf{c})$ is the perturbation function; $\mathbf{c}$ is the dimensionless absolute velocity of the gas molecules. For molecules incident on the channel wall at the point $\mathbf{r}_w$, we have $\mathbf{c} = (c_n, c_\varphi, c_z)$, where $c_n = \mathbf{c}\cdot\mathbf{n}(\mathbf{r}_w)$, and $\mathbf{n}(\mathbf{r}_w)$ is the unit normal to the surface of the channel at the point $\mathbf{r}_w$ and is directed inward, while c_z is the velocity of the molecules along the z-axis.

2. We solve the problem using the kinetic equation with the collision integral in the S-model [3, 4]. Applying the well-known path-integral procedure to this equation [5-7], we obtain an expression for the perturbation function

$$h(s, \mathbf{c}) = \frac{c_z}{c_\perp}\Pi(s, \mathbf{c}) + h^+(0, \mathbf{c})\exp\left(-\frac{\delta s}{c_\perp}\right). \tag{2.1}$$

Here $\Pi(x, \mathbf{c}) = \int\limits_0^x \left\{2\delta u(s') + \frac{4\delta}{15}\left(c^2 - \frac{5}{2}\right)q(s') - \left[\nu + \tau\left(c^2 - \frac{5}{2}\right)\right]\right\} \times \exp\left\{-\frac{\delta(x - s')}{c_\perp}\right\} ds'$; $\mathbf{c}_\perp = (c_n, c_\varphi, 0)$; u is the macroscopic velocity of the gas; q is the heat flux density along the capillary; δ is the rarefaction parameter; h^+ is the perturbation function of molecules reflected from the channel wall; s' is the coordinate measured along the ray $\boldsymbol{\omega} = \mathbf{c}_\perp/c_\perp$ (Fig. 1). Putting $s = \ell$ ($\ell = 2(\mathbf{c}_\perp\cdot\mathbf{n}(0))/c_\perp$) in (2.1), we find a relation between the perturbation function of the molecules incident on the wall and that of the molecules reflected from the wall:

$$h^-(l, \mathbf{c}) = \frac{c_z}{c_\perp}\Pi(l, \mathbf{c}) + h^+(0, \mathbf{c})\exp\left(-\frac{\delta l}{c_\perp}\right). \tag{2.2}$$

A consequence of the axial symmetry of the problem is the relation

$$h^-(l, \mathbf{c}) = h^-(0, \mathbf{c}_R), \tag{2.3}$$

where $\mathbf{c}_R = (-c_n, c_\varphi, c_z)$ is the velocity vector of a gas molecule specularly reflected from the surface of the channel at the point s = 0. The boundary condition is formulated in terms of the scattering kernel. Assuming that the scattering kernel is isotropic, the boundary condition can be written in the form [1]

$$h^+(\mathbf{r}_w, \mathbf{c}) = \widehat{W}h^-(\mathbf{r}_w, \mathbf{c}_R) = \theta[\mathbf{c}\cdot\mathbf{n}(\mathbf{r}_w)]\int d\mathbf{c}'\theta[\mathbf{c}'\cdot\mathbf{n}(\mathbf{r}_w)]\,W_0(\mathbf{c}_R\to\mathbf{c}')\,h^-(\mathbf{r}_w, \mathbf{c}'_R). \tag{2.4}$$

Here $\mathbf{r}_w$ is the position vector of the channel wall, $\theta(x)$ is the Heaviside function, and $W_0(\mathbf{c}_R\to\mathbf{c}')$ is the scattering kernel.

Using (2.2) through (2.4), it is a simple matter to obtain an integral equation for the perturbation function of the molecules reflected from the wall of the channel

$$h^+(0, \mathbf{c}) = \theta(c_n)\int d\mathbf{c}'\theta(c'_n)\,W_0(\mathbf{c}_R\to\mathbf{c}')\left\{h^+(0, \mathbf{c}')\exp\left(-\frac{\delta l}{c'_\perp}\right) + \frac{c'_z}{c'_\perp}\Pi(l, \mathbf{c}')\right\}, \tag{2.5}$$

which is a Fredholm integral equation of the second kind. It was shown in [6] that its solution can be represented as a Neumann series. After substituting this solution into (2.1)

$$h(s, \mathbf{c}) = \widehat{L}c_z\left\{2\delta u + q\frac{4\delta}{15}\left(c^2 - \frac{5}{2}\right) - \left[\nu + \tau\left(c^2 - \frac{5}{2}\right)\right]\right\}, \tag{2.6}$$

where $\widehat{L} = \left\{\widehat{U}_s + \widehat{U}_s\widehat{E}\sum\limits_{k=1}^{\infty}(\widehat{W}\widehat{U}_l\widehat{E})^k\,\widehat{U}_l\right\}$; $\widehat{U}_x\varphi(x, \mathbf{c}) = \frac{1}{c_\perp}\int\limits_0^x ds'\exp\left\{-\frac{\delta(x - s')}{c_\perp}\right\}\times\varphi(s', \mathbf{c})$; $\hat{E} = c_\perp\delta_D(s)$; $\delta_D(x)$ is the Dirac delta-function.

With the help of (2.6) one can construct a system of equations for the unknowns u(s) and q(s) in the Hilbert spaces G_1 and G_2, where the scalar products are defined as $\langle g(r_\perp, \mathbf{c}), f(r_\perp, \mathbf{c})\rangle = \frac{1}{\pi^{3/2}} \int_0^\infty \int_{-\infty}^\infty \int_0^{2\pi} \exp\left(-c_\perp^2 - c_z^2\right) g(r_\perp, \mathbf{c}) f(r_\perp, \mathbf{c}) c_\perp \, dc_\perp dc_z d\vartheta$ for G_1 and $(\langle g(r_\perp, \mathbf{c}), f(r_\perp, \mathbf{c})\rangle) = 2\pi \int_0^1 \langle g(r_\perp, \mathbf{c}), f(r_\perp, \mathbf{c})\rangle r_\perp dr_\perp$ for G_2. This system of equations has the form

$$u = \left\langle c_z, \hat{L}c_z\left\{2\delta u + \frac{4\delta}{15}\left(c^2 - \frac{5}{2}\right) q - \left[\nu + \tau\left(c^2 - \frac{5}{2}\right)\right]\right\}\right\rangle, \tag{2.7}$$

$$q = \left\langle c_z\left(c^2 - \frac{5}{2}\right), \hat{L}c_z\left\{2\delta u + \frac{4\delta}{15}\left(c^2 - \frac{5}{2}\right) q - \left[\nu + \tau\left(c^2 - \frac{5}{2}\right)\right]\right\}\right\rangle.$$

3. The transport integral equation in the system (2.7) is of the Fredholm type. Hence it can be solved using the Bubnov–Galerkin approximation [8]. From the geometry of the problem and the symmetry property (2.3) for h^-, we can choose as base functions $\psi_i(r_\perp) = r_\perp^{2(i-1)}$, i = 1, 2, ... We then use the following approximations for u and q

$$u' = A_1\psi_1 + A_2\psi_2, \quad q' = A_3\psi_1 \tag{3.1}$$

where the A_m are coefficients. Substituting the trial functions (3.1) into (2.7), we find a system of errors characterizing the difference between the approximate solution and the exact one. According to the Bubnov–Galerkin method, the variational constants are calculated from the orthogonality condition of the errors to the base functions. Then we obtain the following system of equations for the coefficients A_m

$$b_n = \sum_{m=1}^{3} a_{nm}A_m, \quad n = 1, 2, 3, \quad a_{ij} = 2\delta\left(s_{ij}^{pp} + l_{ij}^{pp}\right) - \psi_{ij}, \tag{3.2}$$

$$a_{3i} = 2\delta\left(s_{1i}^{Tp} + l_{1i}^{Tp}\right), \quad a_{i3} = \frac{4\delta}{15}\left(s_{i1}^{pT} + l_{i1}^{pT}\right), \quad a_{33} = \frac{4\delta}{15}\left(s_{11}^{TT} + l_{11}^{TT}\right),$$

$$b_i = \nu\left(s_{i1}^{pp} + l_{i1}^{pp}\right) + \tau\left(s_{i1}^{pT} + l_{i1}^{pT}\right), \quad b_3 = \nu\left(s_{11}^{Tp} + l_{11}^{Tp}\right) + \tau\left(s_{11}^{TT} + l_{11}^{TT}\right).$$

Here i, j = 1, 2; $\psi_{ij} = (\langle\psi_i, \psi_j\rangle)$; $s_{ij}^{pp} = (\langle\psi_i c_z, \hat{U}_s c_z \psi_j\rangle)$; $s_{ij}^{pT} = (\langle\psi_i c_z, \hat{U}_s c_z(c^2-5/2)\psi_j\rangle)$; $s_{ij}^{Tp} = (\langle\psi i c_z(c^2-5/2), \hat{U}_s c_z \psi_j\rangle)$; $s_{ij}^{TT} = (\langle\psi_i c_z(c^2-5/2), \hat{U}_s c_z(c^2-5/2)\psi_j\rangle)$;

$$l_{ij}^{pp} = \left(\left\langle \psi_i c_z, \hat{U}_s\hat{E} \sum_{k=1}^{\infty} (\hat{W}\hat{U}_l\hat{E})^k \hat{U}_l c_z \psi_j \right\rangle\right);$$

$$l_{ij}^{pT} = \left(\left\langle \psi_i c_z, \hat{U}_s\hat{E} \sum_{k=1}^{\infty} (\hat{W}\hat{U}_l\hat{E})^k \hat{U}_l c_z (c^2 - 5/2) \psi_j \right\rangle\right);$$

$$l_{ij}^{Tp} = \left(\left\langle \psi_i c_z (c^2 - 5/2), \hat{U}_s\hat{E} \sum_{k=1}^{\infty} (\hat{W}\hat{U}_l\hat{E})^k \hat{U}_l c_z \psi_j \right\rangle\right);$$

$$l_{ij}^{TT} = \left(\left\langle \psi_i c_z (c^2 - 5/2), \hat{U}_s\hat{E} \sum_{k=1}^{\infty} (\hat{W}\hat{U}_l\hat{E})^k \hat{U}_l c_z (c^2 - 5/2) \psi_j \right\rangle\right).$$

The perturbation function h is conveniently written as $h = \nu h_p + \tau h_T$. Then the macroscopic velocity and heat flux density are

$$u' = \nu u_p' + \tau u_T', \quad q' = \nu q_p' + \tau q_T'. \tag{3.3}$$

If the constants A_m and also the elements of the column of free terms are written as $A_m = \nu A_m^p + \tau A_m^T$, $b_n = \nu b_n^p + \tau b_n^T$, then (3.2) splits up into two independent systems of algebraic equations. Solving these equations and substituting the results for the coefficients A_m into (3.3), we have

$$u' = \Delta^{-1}\left\{\nu\left(\Delta_1^p + \Delta_2^p r_\perp^2\right) + \tau\left(\Delta_1^T + \Delta_2^T r_\perp^2\right)\right\}, \quad q' = \Delta^{-1}\left\{\nu\Delta_3^p + \tau\Delta_3^T\right\},$$

where Δ is the determinant of the matrix $[a_{nm}]$; Δ_j^k is the determinant of the matrix obtained by replacing the mth column of $[a_{nm}]$ by the column of free terms $[b_n^k]$, where k = p, T. The resulting expression for the distribution function is ill-suited for the calculation of local values of the macroscopic parameters of the gas motion in the capillary. However this expression is useful in giving a sufficiently accurate description of the behavior of the system as a whole, particularly the integrated characteristics of the motion of the gas in the capillary.

4. The particle and heat fluxes through a cross section of the channel are given by

$$J_N = \int\limits_{\Sigma_\perp} d\mathbf{r}_\perp \int c_z f^0 h(\mathbf{r}_\perp, \mathbf{c})\, d\mathbf{c}, \quad J_q = \int\limits_{\Sigma_\perp} d\mathbf{r}_\perp \int c_z \left(c^2 - \frac{5}{2}\right) f^0 h(\mathbf{r}_\perp, \mathbf{c})\, d\mathbf{c}$$

($\Sigma_\perp$ is the cross section of the channel). Substituting the variational solution found above for $h(\mathbf{r}_\perp, \mathbf{c})$ into this formula, the fluxes transform to

$$J_N = \pi\Delta^{-1}\left\{\nu\left(\Delta_1^p + \frac{\Delta_2^p}{2}\right) + \tau\left(\Delta_1^T + \frac{\Delta_2^T}{2}\right)\right\}, \quad J_q = \pi\Delta^{-1}\{\nu\Delta_3^p + \tau\Delta_3^T\}. \tag{4.1}$$

In the case of free-molecular flow, the fluxes in the linearized problem can be written as

$$J_N = -\frac{4\sqrt{\pi}}{3}\left\{\nu - \frac{\tau}{2} + 3\langle h_0, h^+\rangle^*\right\}, \quad J_q = -\frac{4\sqrt{\pi}}{3}\left\{-\frac{\nu}{2} + \frac{9}{4}\tau + 3\left\langle h_0\left(c^2 - \frac{5}{2}\right), h^+\right\rangle^*\right\},$$

$$\langle f(\mathbf{c}), g(\mathbf{c})\rangle^* = 2\pi^{-1}\int \theta(c_n)\, c_n \exp(-c^2)\, g(\mathbf{c})\, f(\mathbf{c})\, d\mathbf{c}.$$

Substituting in these expressions the solution of (2.5) for the Knudsen flow regime $h^+ = \sum\limits_{k=1}^{\infty} \widehat{W}^k h_0\left[\nu + \tau\left(c^2 - \frac{5}{2}\right)\right]$, $h_0 = 2c_n c_z / c_\perp^2$, we have

$$J_N = -\frac{4\sqrt{\pi}}{3}\left\{\nu\left[1 + 3\sum_{k=1}^{\infty} N_k\right] - \tau\left[\frac{1}{2} - 3\sum_{k=1}^{\infty}\left(M_k - \frac{5}{2}N_k\right)\right]\right\}, \tag{4.2}$$

$$J_q = \frac{4\sqrt{\pi}}{3}\left\{\nu\left[\frac{1}{2} - 3\sum_{k=1}^{\infty}\left(M_k - \frac{5}{2}N_k\right)\right] - \tau\left[\frac{9}{4} + 3\sum_{k=1}^{\infty}\left(\frac{25}{4}N_k - 5M_k + V_k\right)\right]\right\},$$

$$N_k = \langle h_0, \widehat{W}^k h_0\rangle^*, \; M_k = \langle h_0 c^2, \widehat{W}^k h_0\rangle^* = \langle h_0, \widehat{W}^k h_0 c^2\rangle^*,$$

$$V_k = \langle h_0 c^2, \widehat{W}^k h_0 c^2\rangle^*.$$

In the derivation of these expressions, we used the relation $\langle f, \hat{W}^k g\rangle^* = \langle g, \hat{W}^k f\rangle^*$, which is valid if the scattering kernel satisfies the reciprocity relation. The series in (4.2) converge when the reflection of gas molecules from the channel wall is nonspecular. Thus we have obtained expressions for the heat and mass fluxes as functions of the scattering kernel. By comparing the calculations for different models of the scattering kernel with the experimental data, we can estimate the effectiveness of the different models in describing the interaction of the gas molecules with the solid surface, and we can compute the parameters of this interaction.

One method of testing the correctness of our solution is to see if the Onsager relations are satisfied. According to linear nonequilibrium thermodynamics, there exists the following relation between "fluxes" and "forces" of the same tensor rank:

$$J_i = \sum_j L_{ij} x_j \tag{4.3}$$

(L_{ij} are the kinetic coefficients). Hence if we choose the "thermodynamic forces" in the form $x_N = -\nu$, $x_q = -\tau$, then we have from (4.1) and (4.3)

$$L_{NN} = -\pi\Delta^{-1}\left(\Delta_1^p + \frac{\Delta_2^p}{2}\right), \quad L_{Nq} = -\pi\Delta^{-1}\left(\Delta_1^T + \frac{\Delta_2^T}{2}\right), \tag{4.4}$$

$$L_{qN} = -\pi\frac{\Delta_3^p}{\Delta}, \quad L_{qq} = -\pi\Delta^{-1}\Delta_3^T.$$

To prove the equality of the cross kinetic coefficients L_{Nq} and L_{qN}, it is sufficient to expand the determinants Δ_1^T, Δ_2^T, Δ_3^p. In doing this it is necessary to use the self-adjoint property of the operators $\hat{U}_s$ and $\hat{W}$ in the space G_2, which is a consequence of the symmetry of the kernel of the operator $\hat{U}_s$ and the reciprocity of the scattering kernel $W_0(\mathbf{c}' \to \mathbf{c})$. The rest of the proof is obvious, as is the proof of the Onsager relation in the case of free-molecular motion of the gas. Hence we conclude that the method of solution given above does not violate general physical principles.

5. The kinetic coefficients (4.4) characterize the nonisothermal motion of a gas in an infinitely long channel. To calculate these coefficients as functions of the scattering

kernel, we must choose a particular model of the scattering kernel. We use the Cercignani-Lampis model [2] for this purpose. In (4.4) only the ℓ_{ij}^{mn} remain undetermined. It is not possible to obtain analytical formulas for these quantities over the entire range of Knudsen numbers. Therefore we calculate only asymptotic expressions for the kinetic coefficients.

In the case when the motion of the gas in the channel is nearly continuum in nature, i.e., when $\delta \to \infty$, we find, after lengthy but straightforward calculations, that the kinetic coefficients take the form

$$L_{NN} = \frac{\pi}{8}\delta\left(1 + \frac{a_1}{\delta} + \frac{a_2}{\delta^2} + \frac{a_3}{\delta^3}\right), \tag{5.1}$$

$$L_{Nq} = L_{qN} = -\frac{3\pi}{16}\frac{t_0}{\delta}\left(1 + \frac{b_1}{\delta} + \frac{b_2}{\delta^2} + \frac{b_3}{\delta^3}\right), \quad L_{qq} = \frac{15}{8}\frac{\pi}{\delta}\left(1 + \frac{c_1}{\delta} + \frac{c_2}{\delta^2} + \frac{c_3}{\delta^3}\right),$$

where

$$a_1 = l_1 + m_1, \quad a_2 = m_1^2 - m_2 + l_1 m_1 - l_2 - 4,$$

$$a_3 = m_1^3 - 2m_1 m_2 - m_3 + l_1(m_1^2 - m_2) - l_2 m_1 + l_3,$$

$$b_1 = t_0^{-1}(t_0 m_1 - t_1), \quad b_2 = t_0^{-1}\{t_0(m_1^2 - m_2) - t_1 m_1 + t_2\},$$

$$b_3 = t_0^{-1}\{t_0(m_1^3 - 2m_1 m_2 + m_3) - t_1(m_1^2 - m_2) + t_2 m_1 + t_3\},$$

$$c_1 = 3(m_1 - d_1), \quad c_2 = 3(m_1^2 - m_2 - d_1 m_1 + d_2),$$

$$c_3 = 3\{m_1^3 - 2m_1 m_2 - m_3 - d_1(m_1^2 - m_2) + d_2 m_1 + d_3\},$$

$$m_1 = (5\sqrt{\pi})^{-1}\{17 + 5\pi + \varepsilon(1 - 5\pi - \alpha_n + 10S) + 2\varepsilon^3\},$$

$$m_2 = (40\pi)^{-1}\{199\pi - 96 + \varepsilon[32 + 131\pi - \alpha_n(32 + 92\pi) - 48S] - \\ - \varepsilon^2[2\pi(5 - 6\alpha_n + \alpha_n^2) - 16(1 - \alpha_n)S] + 16\varepsilon^3(\pi + 4) - 16\varepsilon^4(\pi - 2S)\},$$

$$m_3 = (40\sqrt{\pi})^{-1}\{444 - \varepsilon(254 - 203\alpha_n + 30\pi\sqrt{\alpha_n} - 30S) - \\ - \varepsilon^3[6 + \alpha_n(15\pi - 9) + 3\alpha_n^2 - 30\pi\sqrt{\alpha_n} + 30S] - \\ - 142\varepsilon^3 - 6\varepsilon^4(3 - 2\alpha_n) - 24\varepsilon^6\},$$

$$l_1 = 4\sqrt{\pi}\varepsilon\alpha_t^{-1} + (5\sqrt{\pi})^{-1}\{3 - \varepsilon(1 - \alpha_n) - 2\varepsilon^3\},$$

$$l_2 = (40\alpha_t)^{-1}\{159 + 2\varepsilon(189 - 4\alpha_n + 160S) + 3\varepsilon^2(13 - 8\alpha_n) + 64\varepsilon^4\},$$

$$l_3 = -(40\alpha_t\sqrt{\pi})^{-1}\{200 + 32\pi + \varepsilon[114 - 1344\pi + \alpha_n(61 + 352\pi) + \\ + 60\pi\sqrt{\alpha_n} + 208S] + \varepsilon^2[32\pi - 52 + \alpha_n(58 - 32\pi) + \alpha_n^2(3 + 8\pi) - \\ - 60\pi\sqrt{\alpha_n} + (64\alpha_n - 80)S] - \varepsilon^3(128 - 9\alpha_n + 3\alpha_n^2) - \\ - 4\varepsilon^4(35 - 3\alpha_n + 32S) + 6\varepsilon^5(1 - 2\alpha_n) + 24\varepsilon^6 - 24\varepsilon^7\},$$

$$t_0 = 3 - \varepsilon(1 - \alpha_n), \quad t_1 = 2\pi^{-1/2}\{7 + 2\pi - \varepsilon[2 + 2\pi - \alpha_n(\pi + 2) - 2S] - \varepsilon^3\},$$

$$t_2 = \frac{1}{8}\{225 - \varepsilon(80 - 83\alpha_n - 6\sqrt{\alpha_n}) - \varepsilon^2(1 - \alpha_n)(9 - 6\sqrt{\alpha_n}) - 20\varepsilon^3 + 12\varepsilon^4(1 - \sqrt{\alpha_n})\},$$

$$t_3 = (4\sqrt{\pi})^{-1}\varepsilon\{88 - 109\alpha_n - 15\pi\sqrt{\alpha_n} + 18S + 3\varepsilon[4 - \alpha_n(6 + \pi) + \\ + 2\alpha_n^2 + \pi\sqrt{\alpha_n}(1 + \alpha_n) + 2(1 - 2\alpha_n)S] - 118\varepsilon^2 - 3\varepsilon^3(7 - 8\alpha_n - 4\pi\sqrt{\alpha_n} + 8S) + 39\varepsilon^5\},$$

$$d_1 = \pi^{-1/2}\{4 + \pi - \varepsilon(\pi - 2S)\}, \quad d_2 = \frac{1}{8}\{45 + \varepsilon(19 - 16\alpha_n)\},$$

$$d_3 = (8\sqrt{\pi})^{-1}\{60 - \varepsilon(44 - 32\alpha_n + 6\pi\sqrt{\alpha_n} - 6S) - \varepsilon^2(3\pi\alpha_n - 6\pi\sqrt{\alpha_n} + 6S) - 16\varepsilon^3\},$$

$$\varepsilon = 1 - \alpha_t, \; S = 2E(\sqrt{1 - \alpha_n}) - \alpha_n K(\sqrt{1 - \alpha_n})$$

(K(x) and E(x) are the complete elliptic integrals of the first and second kinds). Obviously the expansion coefficients a_i, b_i, c_i are functions of the parameters of the model for the scattering kernel. In the case considered here these parameters are α_t and α_n, which can be interpreted as coefficients of accomodation of the tangential component of the momentum and the part of the kinetic energy corresponding to motion along the normal to the channel wall.

The coefficients of accomodation α_n and α_t are related to the physical parameters characterizing the stochastic motion of the gas molecules in the surface force field [1]: $\alpha_n = 1 - \exp(-4d/\ell_n)$, $\alpha_t = 1 - \exp(-2d/\ell_t)$. Here d is the penetration depth; ℓ_n and ℓ_t are two characteristic lengths determined by the diffusion of gas molecules near the surface in velocity space. Concerning the derivation of (5.1), we note that there exists a unique relation between the number of terms in the expansions of the kinetic coefficients in powers of $1/\delta$ and the number of terms in the Neumann series solution of (2.5) needed to obtain the expansion. In derivation of (5.1) it is sufficient to use three terms in the Neumann series for (2.5). The succeeding terms contribute only to the expansion coefficients corresponding to higher powers of $1/\delta$.

The Cercignani–Lampis scattering kernel, which describes the scattering of gas molecules by an anisotropic surface, is a function of three parameters: α_n, α_φ, α_z. The parameters α_φ and α_z can be interpreted as coefficients of accomodation of the tangential components of the momenta of the gas molecules (tangent to the circumference of the circle defining the channel cross section and along the longitudinal axis of the channel, respectively). These coefficients satisfy the inequalities $0 \le \alpha_n \ge 1$, $0 \le \alpha_\varphi \le 2$, $0 \le \alpha_z \le 2$. This model of the scattering kernel leads to the following asymptotic formulas for the kinetic coefficients in the free-molecular regime:

$$L_{NN}=\frac{4\sqrt{\pi}}{3}\left\{1+3\sum_{k=1}^{\infty}N_k\right\},\quad L_{Nq}=L_{qN}=-\frac{4\sqrt{\pi}}{3}\left\{\frac{1}{2}-3\sum_{k=1}^{\infty}\left(M_k-\frac{5}{2}N_k\right)\right\}, \tag{5.2}$$

$$L_{qq}=\frac{4\sqrt{\pi}}{3}\left\{\frac{9}{4}+3\sum_{k=1}^{\infty}\left(\frac{25}{4}N_k-5M_k+V_k\right)\right\}.$$

The functions N_k, M_k, V_k for $\alpha_n = 1$ are

$$N_k=\frac{\pi}{2}\frac{(1-\alpha_z)^k}{(1-\alpha_\varphi)^{2k}}\left\{\ln\left(\frac{1+\sqrt{1-(1-\alpha_\varphi)^{2k}}}{2}\right)+1-\sqrt{1-(1-\alpha_\varphi)^{2k}}\right\}, \tag{5.3}$$

$$M_k=\frac{3}{2}N_k+\frac{\pi}{8}(1-\alpha_z)^k,\quad V_k=\frac{3}{4}N_k[3+2(1-\alpha_z)^{2k}]+\frac{\pi}{2}(1-\alpha_z)^k,$$

and for $\alpha_\varphi = 1$ we have

$$N_k=\frac{(1-\alpha_z)^k}{(1-\alpha_n)^k}\left(p_k-\frac{\pi}{2}\right),\quad M_k=\frac{3}{2}N_k+\frac{(1-\alpha_z)^k}{6(1-\alpha_n)^k}\Phi_k, \tag{5.4}$$

$$V_k=\frac{9}{4}\left\{1+\frac{2}{3}(1-\alpha_z)^{2k}\right\}N_k+\frac{(1-\alpha_z)^k}{4}p_k+\frac{(1-\alpha_z)^k}{2(1-\alpha_n)^k}\Phi_k,$$

$$p_k=2E[(1-\alpha_n)^{k/2}]-[1-(1-\alpha_n)^k]K[(1-\alpha_n)^{k/2}],$$

$$\Phi_k=[1+(1-\alpha_n)^k]E[(1-\alpha_n)^{k/2}]-[1-(1-\alpha_n)^k]K[(1-\alpha_n)^{k/2}].$$

6. Thus we have solved the problem for the transport of heat and mass in the nonisothermal motion of a rarefied gas in a cylindrical capillary. The solution is based on a systematic kinetic treatment of the intermolecular collisions and the gas–surface interaction. General expressions for the kinetic coefficients are found in terms of the interaction of the gas molecules with the surface through the scattering kernel. The results are valid for scattering kernels which are normalized, positive, and which satisfy the reciprocity relation.

The kinetic coefficients are calculated for the viscous and free-molecular flow regimes for the two-parameter Cercignani–Lampis model of the scattering kernel, which describes the stochastic motion of the gas molecules in the surface force field. From (5.1) for the kinetic coefficients in the viscous limit, we obtain the following expressions for the viscous and thermal slip constants

$$\sigma_p=2\left\{\frac{1-\alpha_t}{\alpha_t}+\frac{\alpha_t}{4}+\frac{1}{\pi}+\frac{1-\alpha_t}{\pi}\left[E\left(\sqrt{1-\alpha_n}\right)-\frac{\alpha_n}{2}K\left(\sqrt{1-\alpha_n}\right)\right]\right\}; \tag{6.1}$$

$$\sigma_T=\frac{9}{8}\left\{1-\frac{1}{3}(1-\alpha_n)(1-\alpha_t)\right\}. \tag{6.2}$$

These constants do not depend on the choice of statistical model of the intermolecular collision operator nor on the geometry of the problem, but are determined solely by the gas–surface interaction law. The above results for σ_p and σ_T reduce to results obtained previously in limiting cases. When $\alpha_n = \alpha_t = 1$ and $\alpha_n = \alpha_t = 0$ (which correspond to the cases of diffuse and specular reflection, respectively), they reduce to the values given in [9], and when $\alpha = 1$, they reduce to the results of [10]. In [10] the parameter of α_n in the two-parameter Cercignani–Lampis model of the scattering kernel was unjustifiably assumed to be equal to unity in the calculation of the slip constants. This assumption leads to an expression for σ_T which is independent of the gas–surface interaction parameters, and this contradicts the experimental data [11]. It follows from (6.2) that the thermal slip constant is equal to the corresponding value in the diffuse limit, both in the case of total accomodation of the tangential components of the momentum, and also in the case of total accomodation of the energy for motion along the normal to the wall. Expressions (6.1) and (6.2) can be considered to be a system of equations for the parameters α_n and α_t in terms of the experimental values of the constants σ_p and σ_T. However a graphical solution of this system of equations using the experimental results of [11, 12] shows that the system is inconsistent, i.e., it is impossible to choose a pair of values α_n and α_t which simultaneously characterize the boundary conditions for isothermal and nonisothermal motion of the gas. This means that additional experimental study is needed on the phenomenon of slipping and that better experimental models of the scattering kernel are needed.

If we consider the free-molecular regime of the motion of the gas in the capillary, and assume that $1 - \alpha_z \ll 1$ and $1 - \alpha_\varphi \ll 1$, then it is not difficult to find from (5.2) and (5.3)

$$L_{NN} = \frac{4\sqrt{\pi}}{3}\left\{1 + \frac{3\pi}{8}(1-\alpha_z)\left[1 + (1-\alpha_z) + (1-\alpha_z)^2 + \frac{1}{8}(1-\alpha_\varphi)^2\right]\right\}, \tag{6.3}$$

$$L_{Nq} = L_{qN} = -\frac{2\sqrt{\pi}}{3}\left\{1 + \frac{3\pi}{32}(1-\alpha_z)(1-\alpha_\varphi)^2\right\}, \quad L_{qq} = 3\sqrt{\pi}\left\{1 + \frac{\pi}{4}(1-\alpha_z)^3\left[1 + \frac{(1-\alpha_\varphi)^2}{12(1-\alpha_z)^2}\right]\right\}.$$

Similarly, starting from (5.2) and (5.4), and the assumption that $1 - \alpha_z \ll 1$, $1 - \alpha_n \ll 1$, we obtain

$$L_{NN} = \frac{4\sqrt{\pi}}{3}\left\{1 + \frac{3\pi}{8}(1-\alpha_z)\left[1 + (1-\alpha_z) + \frac{1}{16}(1-\alpha_n)\right]\right\}, \tag{6.4}$$

$$L_{Nq} = L_{qN} = -\frac{2\sqrt{\pi}}{3}\left\{1 + \frac{9\pi}{64}(1-\alpha_z)(1-\alpha_n)\right\},$$

$$L_{qq} = 3\sqrt{\pi}\left\{1 + \frac{3\pi}{32}(1-\alpha_z)(1-\alpha_n)\right\}.$$

The series in (5.2) can be summed when $\alpha_n = 1$ and $\alpha_\varphi = 1$. The kinetic coefficients in this case are given by

$$L_{NN} = \frac{4\sqrt{\pi}}{3}\left\{1 + \frac{3\pi}{8}\frac{1-\alpha_z}{\alpha_z}\right\}, \quad L_{Nq} = L_{qN} = -2\pi/3, \tag{6.5}$$

$$L_{qq} = 3\sqrt{\pi}\left\{1 + \frac{\pi}{4}\frac{(1-\alpha_z)^3}{1-(1-\alpha_z)^3}\right\}.$$

It is evident from (6.3) and (6.4) that the coefficients L_{Nq} and L_{qq} depend more weakly on the parameters of the Cercignani–Lampis model of the scattering kernel than does L_{NN}. According to (6.3) through (6.5), the deciding role in the dependence of Poiseuille flow, thermal creep, and the thermal conductivity on the characteristics of the interaction of the gas with the surface is accomodation of the tangential component of the momentum along the z-axis. For nonisothermal motion of the gas, all three coefficients of accomodation play a comparable role. For example, L_{Nq} reduces to the diffuse value not only when $\alpha_z = 1$, but also when $\alpha_n = \alpha_\varphi = 1$.

For the purpose of verifying the results obtained here, it is of interest to consider the thermomolecular pressure difference effect [13], where a pressure difference arises between two volumes of gas joined by a capillary when the temperatures of the gas in the two volumes are maintained at constant, but unequal values, while the pressures in the vessels are initially identical. In the steady state the thermomolecular pressure difference effect is characterized by an absence of a flux of particles through the cross section of the capillary joining the two volumes. It was shown in [14] that for arbitrary Knudsen number, this effect is described by the expression

$$p_1/p_2 = (T_1/T_2)^\gamma, \tag{6.6}$$

where γ is the thermomolecular pressure difference effect index, and p_i and T_i are the pressures and temperatures in the two volumes after a steady state has been established. The index is given by the relation $\gamma = -(L_{Nq}/L_{NN})$. Using this relation and (6.3) through (6.5), it is possible to compute γ and to determine its dependence on the parameters of the scattering kernel model. Figure 2 shows the dependence of γ on the coefficient α_z for several values of the coefficients α_n, α_φ: $\alpha_n = 0$, $\alpha_\varphi = 1$ (curve 1); $\alpha_n = 1$, $\alpha_\varphi = 0$ (curve 2); $\alpha_n = \alpha_\varphi = 1$ (curve 3). It is seen that as the three parameters α_n, α_φ, and α_z vary from zero to one the index varies from zero to one-half. It is important to note that when $\alpha_z = 0$, the thermomolecular pressure difference effect occurs if α_n or α_φ are nonzero, and does not occur if $\alpha_n = \alpha_\varphi = 1$. Therefore, unlike the diffuse and diffuse-specular reflection models, the Cercignani–Lampis model of the scattering kernel describes the experimentally observed dependence of the thermomolecular pressure difference effect on the nature of the interaction of the gas molecules with the surface of the channel. This fact has been noted before [15].

A detailed analysis of the properties of the Cercignani–Lampis model of the scattering kernel shows that it gives a qualitatively correct description of the experimentally observed dependence of direction of maximum scattered intensity of molecules on the ratio of the temperatures of the surface and the molecular beam. Molecules emitted by a high-temperature source are scattered more nearly tangentially to the solid surface. Therefore if the scattering of the gas by the surface is nondiffuse, the probability of transmission through a channel connecting two bulbs is higher for molecules leaving the "hot" bulb than for molecules leaving the "cold" bulb. Hence the pressure difference between the bulbs is smaller than in the case of diffuse scattering of gas molecules by the walls of the channel. This corresponds (according to (6.6)) to a smaller value of the index γ, and is indicated in Fig. 2.

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